Assignment 4

Hand in no. 6, 7, 8b and 9 by October 10, 2024.

1. Let $C^k[a,b]$ be the vector space of all k-th continuously differentiable functions on $[a,b]$. Show that $\|f\|_k \equiv \sum_{j=0}^k \|f^{(j)}\|_\infty$ defines a norm on $C^k[a,b]$. Furthermore, $f_n \to f$ in $(C[a,b], \|\cdot\|_k)$ means $f_n \rightrightarrows f, \cdots, f_n^{(k)} \rightrightarrows f^{(k)}$.

2. Let $C^\infty[a,b]$ be the vector space of all infinitely many times differentiable functions on $[a,b]$. Show that
$$d_\infty(f,g) = \sum_{k=0}^\infty \frac{1}{2^k} \frac{\|f-g\|_k}{1+\|f-g\|_k}$$
defines a metric on $C^\infty[a,b]$ such that $f_n \to f$ means $\|f_n - f\|_k \to 0$ for all k.

3. In class we showed that the set $P = \{f : \ f(x) > 0, \forall x \in [a,b]\}$ is an open set in $C[a,b]$. Show that it is no longer true if the norm is replaced by the L^1-norm. In other words, for each $f \in P$ and each $\varepsilon > 0$, there is some continuous g which is negative somewhere such that $\|g - f\|_1 < \varepsilon$.

4. Show that $[a,b]$ can be expressed as the intersection of countable open intervals. It shows in particular that countable intersection of open sets may not be open.

5. Optional. Show that every open set in $\mathbb{R}$ can be written as a countable union of disjoint open intervals. Suggestion: Introduce an equivalence relation $x \sim y$ if x and y belongs to the same open interval in the open set and observe that there are at most countable many such intervals.

6. Identify the boundary points, interior points, interior and closure of the following sets in $\mathbb{R}$:
   (a) $[1,2) \cup (2,5) \cup \{10\}$.
   (b) $[0,1] \cap \mathbb{Q}$.
   (c) $\bigcup_{k=1}^\infty (1/(k+1), 1/k)$.
   (d) $\{1,2,3,\cdots\}$.

7. Identify the boundary points, interior points, interior and closure of the following sets in $\mathbb{R}^2$:
   (a) $R \equiv [0,1) \times [2,3) \cup \{0\} \times (3,5)$.
   (b) $\{(x,y) : 1 < x^2 + y^2 \leq 9\}$.
   (c) $\mathbb{R}^2 \setminus \{(1,0), (1/2,0), (1/3,0), (1/4,0), \cdots\}$.

8. Describe the closure and interior of the following sets in $C[0,1]$:
   (a) $\{f : \ f(x) > -1, \ \forall x \in [0,1]\}$.
   (b) $\{f : \ f(0) = f(1)\}$.

9. Find subsets in $\mathbb{R}$ such that $\overline{A \cap B}$ is properly contained in $\overline{A} \cap \overline{B}$.

10. Show that $\overline{E} = \{x \in X : \ d(x,E) = 0\}$ for every non-empty $E \subset X$.